Large-scale anisotropy in scalar turbulence

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The effect of anisotropy on the statistics of a passive tracer transported by a turbulent flow is investigated. We show that under broad conditions an arbitrarily small amount of anisotropy propagates to the large scales where it eventually dominates the structure of the concentration field. This result is obtained analytically in the framework of an exactly solvable model and confirmed by numerical simulations of scalar transport in two-dimensional turbulence.

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The emergence of large-scale anisotropy arising from small-scale sources is a phenomenon that spans the most diverse fields of physics. For instance, the microscopic anisotropy of crystals in mantle rocks in the Earth's interior is believed to induce large-scale seismic anisotropy [1], and some small anisotropic perturbation in an early cosmological era, evolving through gravitational collapse, is thought to be responsible for the large-scale anisotropy of the cosmic microwave background radiation [2]. Conversely, in statistical physics, microscopic details such as lattice anisotropies may be wiped out by the dynamics allowing a recovery of symmetry and universality at large scales [3]. In the theoretical and experimental analysis of turbulence much attention has been devoted to the anisotropy of the fine scales of fluid motion (see, e.g., [4] and references therein). Here, we take a different viewpoint and investigate the effect of anisotropy on the large-scale statistics of turbulence. In this Letter we show how, unexpectedly, breaking rotational invariance by an arbitrarily small amount at a given scale induces a strong anisotropy on the large scales, and symmetry is never restored.

We consider the evolution of a passive tracer described by a concentration field $\theta(\boldsymbol{x}, t)$ and transported by a turbulent flow $\boldsymbol{v}(\boldsymbol{x}, t)$

$$\partial_t \theta + \boldsymbol{v} \cdot \boldsymbol{\nabla} \theta = \kappa \Delta \theta + f, \tag{1}$$

where $\boldsymbol{v}$ is an incompressible, statistically homogeneous and isotropic velocity field. The external driving f is the source of scalar field fluctuations acting at a characteristic scale l_f. The turbulent cascade toward small scales produces fine-scale structures of concentration that are eventually smeared out by diffusion at scales $r_d \ll l_f$, resulting in a statistically stationary state where input and dissipation are in balance on average. The pumping mechanism can be chosen so as to introduce a certain degree of anisotropy, that propagates across scales and may in principle pervade the system. However, the disordered motion of fluid particles induced by the underlying turbulent, isotropic medium, might be sufficient to restore rotational invariance at scales far below or above l_f. Indeed, this is the case at small scales $r \ll l_f$, where it can be shown that the dominant contribution to the statistics of the scalar field θ is isotropic [4]. At large scales $r \gg l_f$, since no upscale cascade of scalar fluctuations occurs, *a fortiori* one would expect an essentially isotropic concentration field. On the contrary, here we give theoretical and numerical evidence that large-scale statistics is dominated by the anisotropic contribution under very broad conditions. We show that: (*i*) The correlation function $\langle \theta(\boldsymbol{r}, t)\theta(\boldsymbol{0}, t)\rangle$ at scales $r \gg l_f$ is dominated by its anisotropic component decaying as a power law with an anomalous scaling exponent, as opposed to the exponential fall-off of the isotropic part. This result is obtained analytically in the framework of the exactly solvable Kraichnan model and its validity for realistic flows is demonstrated by numerical simulations of passive scalar advection in the inverse cascade of two-dimensional turbulence. (*ii*) Large-scale anisotropy manifests itself in the concentration field with the appearance of "pearl necklace" structures aligned with the preferential direction imposed by the microscopic anisotropy. (*iii*) The loss of isotropy at large scales can be interpreted as a breakdown of equilibrium Gibbs statistics for the anisotropic degrees of freedom; (*iv*) In the Lagrangian interpretation of passive scalar transport the emergence of anisotropic power-law decay of correlation is associated to a long-lasting memory of the initial orientation of particle pairs advected by the flow.

Let us first consider the Kraichnan model of passive scalar advection (see, e.g., [5] for a thorough review), where $\boldsymbol{v}$ is a Gaussian, self-similar, incompressible, statistically homogeneous and isotropic, white-in-time, $d-$dimensional velocity field. Its statistics is characterized by the correlation $S_{\alpha\beta}(\boldsymbol{r})\delta(t) = \langle [v_\alpha(\boldsymbol{r}, t) - v_\alpha(\boldsymbol{0}, t)][v_\beta(\boldsymbol{r}, 0) - v_\beta(\boldsymbol{0}, 0)]\rangle = 2Dr^\xi[(d + \xi - 1)\delta_{\alpha\beta} - \xi\, r_\alpha r_\beta/r^2]\delta(t)$. The exponent ξ measures the degree of roughness of the velocity field and lies in the range $0 < \xi < 2$, the two extremes corresponding to Brownian diffusion and smooth velocity, respectively. The assumption of δ-correlation is of course far from being realistic, yet it has the remarkable advantage of leading to closed

equations for equal-time correlation functions. In the following it will be sufficient to focus on the two-point correlation function $C(\boldsymbol{r}) = \langle \theta(\boldsymbol{r},t)\theta(\boldsymbol{0},t)\rangle$. In the limit of vanishing diffusivity $\kappa \to 0$ and in the statistically stationary state, C obeys the partial differential equation $\mathcal{M}C(\boldsymbol{r}) = -F(\boldsymbol{r})$ where $\mathcal{M} = \frac{1}{2}S_{\alpha\beta}(\boldsymbol{r})\frac{\partial}{\partial r_\alpha}\frac{\partial}{\partial r_\beta}$. Here F is the correlation function of the Gaussian, white-in-time, statistically stationary, homogeneous, anisotropic forcing $\langle f(\boldsymbol{r},t)f(\boldsymbol{0},0)\rangle = F(\boldsymbol{r})\delta(t)$. At scales $r \lesssim l_f$ it equals the average input rate of scalar and then decays rapidly to zero, e.g. exponentially, as $r \gg l_f$. By virtue of the statistical isotropy of the velocity field, the operator $\mathcal{M}$ assumes a particularly simple form in radial coordinates: $\mathcal{M} = D[(d-1)r^{1-d}\partial_r r^{d-1+\xi}\partial_r + (d+\xi-1)r^{\xi-2}\mathcal{L}^2]$ where $\mathcal{L}^2$ is the d-dimensional squared angular momentum operator. It is then convenient to decompose the correlation functions on a basis of eigenfunctions of angular momentum $\mathcal{L}^2 Y_j = -j(j+d-2)Y_j$ with positive integer j. The short-hand notation $Y_j(\hat{\boldsymbol{r}})$, where $\hat{\boldsymbol{r}} = \boldsymbol{r}/r$, does not account for degeneracies and stands for the trigonometric functions in $d=2$ and the spherical harmonics in $d=3$. Accordingly, we define the components of the correlation functions in the j-th anisotropic sector as $C(\boldsymbol{r}) = \sum_j C_j(r)Y_j(\hat{\boldsymbol{r}})$ and similarly for $F(\boldsymbol{r})$, where C_j and F_j depend on $r = |\boldsymbol{r}|$ only. This yields a system of uncoupled differential equations in the radial variable for each anisotropic component $\mathcal{M}_j C_j(r) = -F_j(r)$, where $\mathcal{M}_j = D[(d-1)r^{1-d}\frac{d}{dr}r^{d-1+\xi}\frac{d}{dr} - j(j+d-2)(d+\xi-1)r^{\xi-2}]$, that can be solved in each sector j. The resulting $C_j(r)$ is a linear combination of a particular solution determined by $F_j(r)$, and a homogeneous one $Z_j(r)$, a "zero mode". It is easy to see that the former behaves as $r^{2-\xi+j}$ for $r \ll l_f$ (recall that $F_j \sim r^j$ at small r if F is analytic in the neighborhood of $r=0$) and that it must fall off exponentially fast for $r \gg l_f$, as dictated by the decay of F_j. The homogeneous solutions are $Z_j^\pm(r) = r^{\zeta_j^\pm}$ with scaling exponents $\zeta_j^\pm = \frac{1}{2}[-d+2-\xi \pm \sqrt{(d-2+\xi)^2 + 4\frac{j(j+d-2)(d+\xi-1)}{d-1}}]$. The zero mode with positive scaling exponent ζ_j^+ appears at small scales whereas the zero mode with negative scaling exponent ζ_j^- is relevant in the range $r \gg l_f$. In order to fully characterize the large-scale behavior of the correlation function, it is necessary to identify the prefactor appearing in front of the homogeneous solution. This can be accomplished by writing the equation for C_j in integral form: $C_j(r) = \int_0^\infty G_j(r,\rho)F_j(\rho)d\rho$, where $G_j(r,\rho)$ is the kernel of $-\mathcal{M}_j^{-1}$, i.e. the solution of $\mathcal{M}_j G_j(r,\rho) = -\delta(r-\rho)$. The explicit form is $G_j(r,\rho) = A(\rho)Z_+^j(r)Z_-^j(\rho)$ for $r < \rho$ and $G_j(r,\rho) = A(\rho)Z_-^j(r)Z_+^j(\rho)$ for $r > \rho$, with $A(\rho) = \rho^{d-1}/[D(\zeta_j^+ - \zeta_j^-)]$. Plugging this expression in the integral form for the correlation function yields a large-scale behavior $C_j(r) \approx Q_j Z_j^-(r)$ + exponentially decaying terms. The quantity $Q_j = \int_0^\infty A(\rho)Z_+^j(\rho)F_j(\rho)\,d\rho$ is of crucial importance and plays the role of a "charge"

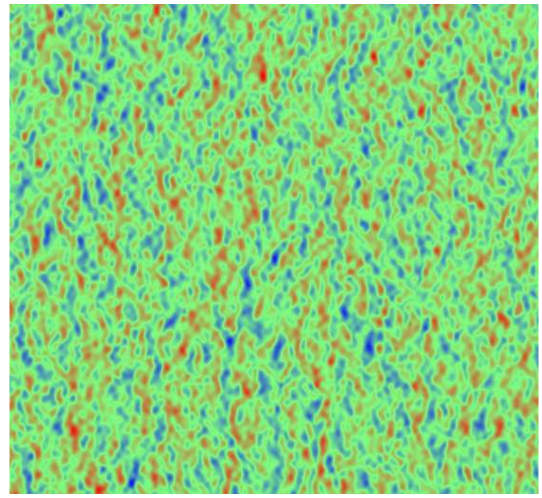

FIG. 1: Image of a scalar field for the Kraichnan model at $\xi = 0$, $d = 2$, $Q_0 = 0$, $Q_2 \neq 0$. The width of a single scalar patch is $\sim l_f$. Forcing is preferentially acting in the vertical direction. Here $C_0(r) \sim e^{-r^2/l_f^2}$ and $C_2(r) \sim r^{-2}$ at $r \gg l_f$.

in analogy with electrostatics [6]. In the isotropic sector $j=0$, it reduces to $Q_0 = \frac{\Gamma(d/2)}{2\pi^{d/2}D(d-2+\xi)}\int F(\boldsymbol{r})\,d\boldsymbol{r}$. When the isotropic charge $Q_0 \neq 0$, the leading behavior at large scales is isotropic, $C(\boldsymbol{r}) \sim C_0(r) \sim Q_0 r^{-d+2-\xi}$. The most interesting situation is when $Q_0 = 0$, corresponding to the broad class of forcings localized in wavenumber space ($Q_0 \propto \hat{F}(\boldsymbol{k}=\boldsymbol{0})$). In this event, there is no power-law contribution from the isotropic zero mode and therefore the isotropic part of the correlation function is characterized by an exponential decay at large r [6, 7]. In the anisotropic sectors, it appears immediately that there is no reason to expect a null charge and the generic situation is $Q_j \neq 0$ for $j > 0$ (see [8]). As a result, *the large-scale correlation is dominated by the anisotropic contribution* arising from the zero-mode, $C(\boldsymbol{r}) \sim r^{\zeta_j^-}Y_j(\hat{\boldsymbol{r}})$, that largely overweights the exponentially small isotropic part. Among the various contributions arising from different sectors, the leading one corresponds to the lowest nonzero j excited by the forcing, typically $j=2$ (odd j's are switched on only by breaking reflection invariance).

For the sake of illustration, we show in Fig. 1 an instance of a scalar field corresponding to the simple case of Kraichnan advection by a very rough velocity ($\xi = 0$), in $d = 2$ and with $Q_0 = 0$. Large-scale anisotropy manifests itself in the appearance of "pearl necklaces" made of like-sign scalar patches of size $\sim l_f$. These are aligned along the preferred direction of the forcing and extend for a length $\gg l_f$.

It is worth pointing out the relationship between the appearance of anisotropic, anomalous scaling in the large-scale behavior of the scalar correlation function and equilibrium statistics. At large scales, because of the absence of scalar flux, the system could be expected to be in

equilibrium and obey Gibbs statistics. In physical space this corresponds to a concentration field organized in independently distributed scalar patches of size l_f. As recently shown in Refs. [6, 7], this is not true and substantial deviations are observed at the level of multipoint correlation functions already in the isotropic case. This departure has been traced back to the existence of nontrivial zero-modes in that case as well. In the present case the breakdown of Gibbs statistics has an even more dramatic manifestation since it occurs already for two-point correlation functions, i.e. at the level of the spectral distribution of concentration. We now rephrase the previous findings in terms of the averaged scalar spectral density, i.e. the Fourier transform of the correlation function, $\hat{C}(\boldsymbol{k}) = \langle|\hat{\theta}(\boldsymbol{k},t)|^2\rangle$, and its decomposition in angular sectors in wavenumber space $\sum_j \hat{C}_j(k)Y_j(\hat{\boldsymbol{k}})$, where $\hat{\boldsymbol{k}} = \boldsymbol{k}/k$. For a correlation function decaying exponentially to zero at large r, representative of large-scale equipartition in physical space, the spectral density is analytic in a neighborhood of $k = 0$. In the series for $\hat{C}(\boldsymbol{k})$ the harmonic $Y_j(\hat{\boldsymbol{k}})$ appears only in the powers of $\boldsymbol{k}$ of order $\geq j$, yielding the long wavelength behavior $\hat{C}_j^{(eq)}(k) \sim k^j$. This defines the equipartition spectrum for generic anisotropic fluctuations. However, because of the appearance of nontrivial zero modes in the anisotropic sectors the actual spectral density contains also a contribution $\hat{C}_j^{(zero)}(k) \sim k^{-d-\zeta_j^-}$ that is responsible for the power-law decay of correlations in physical space with $j > 0$. For the Kolmogorov-Richardson value $\xi = 4/3$ and the sector $j = 2$ the anomalous spectrum always dominates the equipartition contribution in spectral space as well ($-d - \zeta_{j=2}^- < 2$ for all $\xi < 3/2$).

It is useful to reinterpret the results obtained so far within the framework of the Lagrangian interpretation of passive scalar transport. The correlation function can be generically written as $C(\boldsymbol{r}) = \int T(\boldsymbol{\rho}|\boldsymbol{r})\,F(\boldsymbol{\rho})\,d\boldsymbol{\rho}$ where $T(\boldsymbol{\rho}|\boldsymbol{r})\,d\boldsymbol{\rho}$ is the average time spent at a separation $\boldsymbol{\rho}+d\boldsymbol{\rho}$ by a pair of particles that end their trajectories at a separation $\boldsymbol{r}$. In the Kraichnan model T is the kernel of the operator $-\mathcal{M}^{-1}$. Since the action of the forcing is restricted to scales $\sim l_f$, the large-scale behavior of the correlation function is essentially dominated by the ensemble of trajectories that have spent in the past a sufficiently long time at a short distance $\rho \lesssim l_f \ll r$. When r/ρ tends to infinity T becomes independent of ρ and we obtain $T \sim r^{-d+2-\xi}$ [9]. The dependency on the final orientation $\hat{\boldsymbol{r}}$ is also lost. This leads to the estimate $C(r) \sim Q_0 r^{-d+2-\xi}$. However, as noticed previously, when $Q_0 = 0$ the isotropic part of the correlation function receives only exponentially small contributions from the forcing in the range $\rho \sim r \gg l_f$. Let us now turn our attention to the anisotropic part of the correlation function. Projecting $C(\boldsymbol{r})$ over $Y_j(\hat{\boldsymbol{r}})$ for $j > 0$ amounts to give different weigths, positive and negative, to particle pairs oriented in different directions $\hat{\boldsymbol{r}}$. Therefore C_j can be interpreted as a difference of times spent at $\rho \lesssim l_f$ by differently oriented pairs. The first key point is that the trajectories preserve a long-lasting memory of their initial orientation, with a slow power-law decay in r that reflects in the behavior of the correlation function. Indeed, it can be shown that in the Kraichnan model $T(\boldsymbol{\rho}|\boldsymbol{r}) = \sum_j b_j r^{\zeta_j^-} \rho^{\zeta_j^+} Y_j(\hat{\boldsymbol{r}})Y_j(\hat{\boldsymbol{\rho}})$ for $r > \rho$. Plugging this expression in the integral form of the correlation function gives the result $C_j(r) \sim Q_j r^{\zeta_j^-}$ as above. Here emerges the second important point, i.e. the dependence of T on ρ: differently oriented trajectories sample the forcing unevenly in scales as $\rho \sim l_f$ and this results in a nonvanishing charge Q_j for $j > 0$.

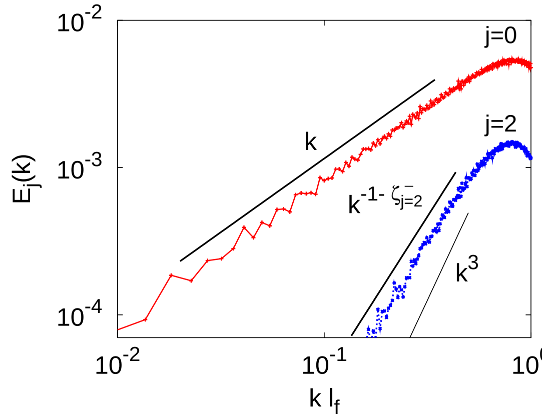

FIG. 2: Simulations of passive scalar advection by two-dimensional Navier-Stokes turbulence in the inverse cascade (for details see [7]). Here are shown the spectra $E_j(k)$ of concentration in the isotropic ($j = 0$) and anisotropic ($j = 2$) sectors. The spectral slopes are 1.0 ± 0.05 and 2.2 ± 0.1, respectively ($\zeta_{j=2}^- = -3.2$). The equilibrium spectra k and k^3 are also shown for comparison.

A remarkable advantage of the Lagrangian interpretation is that it does not make appeal to the special features of the Kraichnan model. This suggests that the same mechanisms are at work for realistic turbulent flows as well, and this expectation has been repeatedly confirmed for different aspects of passive scalar transport [7, 10, 11]. Here we show that anisotropy dominates the large-scale statistics for real flows by showing the results of a numerical investigation of passive scalar transport in the inverse cascade of two-dimensional Navier-Stokes turbulence. This flow has been studied in great detail, both experimentally in fast flowing soap films [12] and in shallow layers of electromagnetically driven electrolyte solutions [13, 14], and numerically [15, 16, 17]. The velocity field $\boldsymbol{v}$ is statistically homogeneous, isotropic, and scale-invariant with exponent $h = 1/3$ ($\delta_r v \sim r^h$) in the range $l_f^v \lesssim r \lesssim L_v$, where l_f^v denotes the kinetic energy injection length and L_v the velocity integral scale. The scalar field is governed by Eq. (1) driven by a homogeneous, anisotropic, Gaussian, δ-correlated driving f that

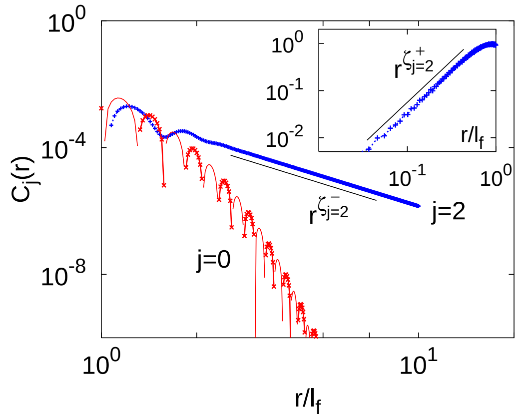

FIG. 3: Main frame: Correlation functions $C_j(r)$ at large scales $r \gtrsim l_f$ for $j = 0$ and $j = 2$. Notice the exponential decay of the isotropic part as opposed to the power-law behavior of the anisotropic component. In order to limit the statistical noise at large values of r/l_f these two curves have been obtained by interpolating the spectra shown in Fig. 2 at small k as power laws k and $k^{-1-\zeta^-_{j=2}}$, respectively, and computing the correlation function from the integral $C_j(r) = \int_0^\infty J_j(kr)E_j(k)/k\,dk$. Here $\zeta^-_{j=2} = -3.2$. Inset: Correlation function $C_{j=2}(r)$ at small scales $r \lesssim l_f$. The solid line is $r^{\zeta^+_{j=2}}$ with $\zeta^+_{j=2} = 1.8$.

excites the sectors $j = 0$ and $j = 2$ and satisfies the condition of null isotropic charge (see [8]). The various lengthscales are ordered as follows: $l^v_f \ll r_d \ll l_f \ll L_v$. In Fig. 2 we show the spectral content of scalar fluctuations $E_j(k) = \pi^{-1}k\int_0^{2\pi}\cos(j\phi_k)\,\hat{C}(\boldsymbol{k})\,d\phi_k = k\hat{C}_j(k)$ at $kl_f < 1$, i.e. at large scales. The isotropic spectrum ($j = 0$) agrees very well with the Gibbs equilibrium distribution, $E_{j=0}(k) \propto k$, and corresponds to exponentially decreasing isotropic correlation at large scales (see the main frame of Fig. 3) in agreement with the theoretical arguments presented above. The anisotropic spectrum shows a power-law behavior $E_{j=2}(k) \sim k^{2.2\pm0.1}$ definitely dominating over the equilibrium spectrum $E^{eq}_{j=2}(k) \sim k^3$. In physical space, this translates in a power-law decay of the anisotropic correlation function $C_{j=2}(r)$, as shown in the main frame of Fig. 3, and leads to the estimate $\zeta^-_{j=2} \simeq -3.2 \pm 0.1$. Therefore, the correlation function at large scales is dominated by the anisotropic power-law decay for 2D Navier-Stokes advection as well. Finally, we notice that for incompressible, time-reversible, self-similar flows the two zero-mode exponents are conjugated by the dimensional relation $\zeta^+_j + \zeta^-_j = -d + 1 - h$ (within the Kraichnan model $1 - h = 2 - \xi$, due to the δ-correlation in time). In the inset of Fig. 3 we show the behavior of the correlation function $C_{j=2}(r) \sim r^{\zeta^+_j}$ at small scales $r \ll l_f$ that indeed displays an exponent $\zeta^+_j = 1.8\pm0.1$ compatible with the previous relation [18].

In summary, we have shown that microscopic anisotropies introduced by the forcing have a dramatic imprint on the large-scale statistics of passive scalar turbulence. From this result new questions arise naturally, the most intriguing one being whether the large scales of hydrodynamic turbulence show such striking properties as well. Further theoretical, experimental and numerical effort in this direction is needed to elucidate this point.

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